1 Higher Inductive Types Via Impredicative Encodings

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The traditional, "impredicative" encodings of finitary inductive types in System F do not satisfy η-equalities. If we try to recycle the same technique in dependent type theory (by postulating an impredicative universe), the lack of η-equality translates into the absence of dependent eliminators.

S. Awodey, J. Frey, and S. Speight [2] refine these encodings in homotopy type theory (HoTT) with an added impredicative universe to recover the proper η-equalities. While they also explore the applicability of their techniques to some higher order inductive types, they only state the completeness of their encodings for set-level $\mathcal{W}$-types. These refinements only eliminate into other h-sets.

M. Shulman [4] sketches a strategy which takes place in traditional HoTT as presented by The Univalent Foundations Program [6] and allows for elimination into types without any homotopy level restrictions. The proposed project builds on the latter piece of literature. Possible research directions include:

1. turning the sketch found in M. Shulman [4] into a rigorous proof;
2. extending this method to higher inductive types, for different definitions of higher inductive types proposed in the literature (see H. Basold, H. Geuvers, and N. van der Weide [3] and K. Sojakova [5]);
3. extending this method to coinductive types;
4. proposing a definition of "higher coinductive types" to which such a strategy can be applied;
5. formalising any subset of these results in a proof assistant such as Agda [1].

References

[1] Agda Community. 2024. Agda. Retrieved from https://wiki.portal.chalmers.se/agda

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